PROPERTY LATTICES FOR INDEPENDENT QUANTUM SYSTEMS

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We consider the description of two independent quantum systems by a complete atomistic ortho-lattice (cao-lattice) $\mathcal{L}$. It is known that since the two systems are independent, no Hilbert space description is possible, *i.e.* $\mathcal{L} \neq \mathrm{P}(\mathcal{H})$, the lattice of closed subspaces of a Hilbert space (theorem 1). We impose five conditions on $\mathcal{L}$. Four of them are shown to be physically necessary. The last one relates the orthogonality between states in each system to the ortho-complementation of $\mathcal{L}$. It can be justified if one assumes that the orthogonality between states in the total system induces the ortho-complementation of $\mathcal{L}$. We prove that if $\mathcal{L}$ satisfies these five conditions, then $\mathcal{L}$ is the separated product proposed by Aerts in 1982 to describe independent quantum systems (theorem 2). Finally, we give strong arguments to exclude the separated product and therefore our last condition. As a consequence, we ask whether among the ca-lattices that satisfy our first four basic necessary conditions, there exists an ortho-complemented one different from the separated product.

Keywords: quantum mechanics, independent systems, property lattices.

1 Motivations and notations

In ordinary quantum mechanics, a system is described by a (separable) Hilbert space over the complex numbers. The state space is given by $\Sigma = \mathcal{H}^* / \mathbb{C}$. Moreover, to any yes-no experiment α on the system corresponds a $\mu(\alpha) \subset \Sigma$ with $\mu(\alpha)^{\perp\perp} = \mu(\alpha)$ (a closed subspace) such that the answer "yes" is certain (*i.e.* the answer "no" is impossible) if and only if the state of the system is in $\mu(\alpha)$. Finally, the map μ is assumed to be surjective.

When two quantum systems are independent, Einstein, Podolsky and Rosen pointed out that no Hilbert space description for the total system is possible [5]. As a consequence the mathematical description in the sense of Birkhoff and von Neumann [3] of that situation appears as a natural question. To this end, we need a generalization of the Hilbert space framework: Let Q the set of all possible yes-no experiments on a system S at a certain time t. Let Σ be a set (the state space) and $\mathcal{L} \subset 2^{\Sigma}$ a set of subsets of Σ such that there is a surjective map $\mu : Q \to \mathcal{L}$ with the property that the answer "yes" for α is certain if and only if the state of S is in $\mu(\alpha)$. Then, following Aerts [1], we will assume that $\mathcal{L}$ is a p-lattice ($\mathcal{L}$ is called the property lattice)

Definition 1 Let Σ be a set and $\mathcal{L} \subset 2^{\Sigma}$. We call $\mathcal{L}$ a p-lattice if
(1) $\emptyset, \Sigma \in \mathcal{L}$,
(2) $\cap a_\alpha \in \mathcal{L}$ for any family of elements of $\mathcal{L}$,
(3) $\{p\} \in \mathcal{L}$, $\forall p \in \Sigma$.

Remark 1 A p-lattice is a complete atomistic lattice (say ca lattice). The set of atoms is given by $\{\{p\};\ p \in \Sigma\}$. A complete atomistic ortho-lattice (say cao-lattice) is ortho-isomorphic to the p-lattice $\{a \subset \Sigma;\ a^{\perp\perp} = a\}$, where Σ is the set of atoms.

(1) Define I the trivial yes-no experiment by: "Do nothing on S and answer yes", and $O = I^{\sim}$, that is I with answers "yes" and "no" inverted. Then clearly $\mu(O) = \emptyset$ and $\mu(I) = \Sigma$. (2) Further, let α_i be a family of yes-no experiments on S. Define $\pi\alpha_i$ by: "choose freely an α_i and perform it". Then $\mu(\pi\alpha_i) = \cap\mu(\alpha_i)$. (3) Finally, for $p \in \Sigma$ define $a_p := \cap\{a \in \mathcal{L};\ p \in a\}$. Then $p \in a_p$ and $\varepsilon_p := \mu^{-1}([a_p, \Sigma])$ is the set of certain yes-no experiments (*i.e.* the answer "yes" is certain) when the state of the system is p. Suppose now that $\{p\} \neq a_p$. Let $p \neq q \in a_p$. Then $\varepsilon_p \subset \varepsilon_q$. We want to assume that when the state of the system changes, some yes-no experiments become certain and some others do not remain certain.

Finally, it is usually assumed that $\mathcal{L}$ has an ortho-complementation. Note that there was an attempt to justify this axiom [1], based on the following natural symmetric anti-reflexive binary relation on Σ:

$$p \perp q \Leftrightarrow \exists\ \alpha \in Q;\ p \in \mu(\alpha) \text{ and } q \in \mu(\alpha^{\sim}) \tag{1}$$

where $\alpha^{\sim}$ is the same yes-no experiment as α but with switched answers. It is in general delicate to give physical arguments for this relation to induce an ortho-complementation on $\mathcal{L}$, see [1].

The time evolution of a system is given by a map $u : \Sigma_{t_0} \to \Sigma_{t_1}$. W. Daniel [4] pointed out that u must satisfy

$$u^{-1}(b) \in \mathcal{L}_{t_0},\ \forall b \in \mathcal{L}_{t_1}\ , \tag{2}$$

since $\mu(\alpha) = u^{-1}(\mu(\beta))$ for any β with $\mu(\beta) = b$, where α is the yes-no experiment on the system at time t_0 defined by: "evolve the system from time t_0 to time t_1 and perform β".

Proposition 1 *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be p-lattices and $f : \Sigma_1 \to \Sigma_2$. Assume that f satisfies condition (2). Then $g(a) := \vee f(a)$ is $\vee$-preserving and equals f on the atoms.*

Proof : Let $\{a_\alpha \in \mathcal{L}_1\}_{\alpha\in\omega}$, then since f satisfies condition (2), we have $\vee a_\alpha \subset f^{-1}(\vee g(a_\alpha))$, that is $g(\vee a_\alpha) \subset \vee g(a_\alpha)$. Moreover, since g preserves the order, $\vee g(a_\alpha) \subset g(\vee a_\alpha)$.

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Let S_1 and S_2 be two physical systems described by two p-lattices $\mathcal{L}_1$ and $\mathcal{L}_2$. Suppose that at a given time t_0, the two systems are independent. This means that at time t_0 any experiment on one system does not alter the state of the other system (and, in particular, that the two systems do not interact at time t_0). It is the case in many experiments, for instance before the interaction begins between two systems prepared in two independent parts of the experimental device. Let $\mathcal{L}_{ind}$ a p-lattice describing the physical properties of the total system S at time t_0 (*i.e.* $\mathcal{L}_{ind}$ is the property lattice of S). Then we will assume that:

Definition 2 Let $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}$ be p-lattices. Denote by $\mathrm{Aut}(\mathcal{L}_i)$ the set of automorphisms of $\mathcal{L}_i$. We say that $\mathcal{L}$ is
(P1) if $\Sigma = \Sigma_1 \times \Sigma_2$,
(P2) if $a_1 \times \Sigma_2 \cup \Sigma_1 \times a_2 \in \mathcal{L}$, $\forall a_i \in \mathcal{L}_i$,
(P3) if $a_1 \times \Sigma_2 \in \mathcal{L} \Rightarrow a_1 \in \mathcal{L}_1$, $\Sigma_1 \times a_2 \in \mathcal{L} \Rightarrow a_2 \in \mathcal{L}_2$,
(P4) if $\exists\, \emptyset \neq W_i \subset \mathrm{Aut}(\mathcal{L}_i);\ \ u_1 \times u_2(a) \in \mathcal{L},\ \forall a \in \mathcal{L},\ u_i \in W_i$.

We now briefly argue why these conditions are necessary (for more details see [7]): (P1) Since S_1 and S_2 are independent, the state of S is a product state. (P2) Let $\alpha_1 \in Q_{S_1}$, then $\alpha_1 \in Q_S$ and since S_1 and S_2 are independent, $\mu(\alpha_1) = \mu_1(\alpha_1) \times \Sigma_2$. Moreover, let $\alpha_2 \in Q_{S_2}$. Perform α_1 then α_2 or α_2 then α_1 or both simultaneously. Denote this experiment as E. It has four possible outcomes: yy, yn, ny and nn. Let $\alpha_1 \times \alpha_2$ be the yes-no experiment on S defined by: "perform E and answer "yes" if one gets yy, yn or ny and "no" if one gets nn". Then, since S_1 and S_2 are independent, $\mu(\alpha_1) \cup \mu(\alpha_2) = \mu(\alpha_1 \times \alpha_2)$. (P3) Let $a_1 \times \Sigma_2 \in \mathcal{L}$ and $\alpha \in \mu^{-1}(a_1 \times \Sigma_2)$. Then the answer "yes" for α is certain if and only if the state p_1 of the first system is in a_1, so that $\alpha \in Q_{S_1}$ and so $a_1 \in \mathcal{L}_1$. (P4) Suppose that S_1 and S_2 evolve from time t_0 to time t_1 without interacting. Then the evolution of the total system is given by $u_1 \times u_2$ where u_i is the evolution map of system i and P4 is equivalent to condition (2). Of course, in general, not any automorphism of $\mathcal{L}_i$ represents a possible evolution of system i. But any automorphism of $\mathcal{L}_i$ can be interpreted as a passive action on system i, and therefore P4 should hold for any automorphism of $\mathcal{L}_i$. Moreover, if $\mathcal{L}_i = \mathrm{P}(\mathcal{H}_i)$ the lattice of closed subspaces of a Hilbert space, we must restrict to unitary maps. Finally, remark that if $u_i \in W_i \Rightarrow u_i^{-1} \in W_i$, then

$$(P4) \Rightarrow u_1 \times u_2 \text{ is an isomorphism of } \mathcal{L},\ \forall u_i \in W_i\ .$$

Assume now that $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}$ are cao-lattices. Note Σ_i (Σ) the atom space of $\mathcal{L}_i$ (of $\mathcal{L}$) and $\perp_i$ ($\perp$) the orthogonality relation on Σ_i (on Σ). Then we assume that
(P5) $p_1 \perp_1 q_1$ or $p_2 \perp_2 q_2 \Rightarrow (p_1, p_2) \perp (q_1, q_2)$.
This assumption comes from relation (1): let $p_1,\ q_1 \in \Sigma_1$ be two orthogonal states of S_1, that is there exists $\alpha \in Q_{S_1}$ such that $p_1 \in \mu(\alpha)$ and $q_1 \in \mu(\alpha^{\sim})$. Let $r_2,\ s_2 \in \Sigma_2$ be two arbitrary states of S_2. Since α is a question on S and S_1 and S_2 are independent, from (1) we ask that $(p_1, r_2) \perp (q_1, s_2)$ for all $r_2,\ s_2 \in \Sigma_2$ and $p_1 \perp_1 q_1 \in \Sigma_1$. But again, it is delicate to give physical arguments for this relation to induce an ortho-complementation on $\mathcal{L}_{ind}$ (see [7]).

Remark 2 Conditions P1 to P5 can easily be generalized for n independent quantum systems. P2 then reads $a_1 \times \Sigma_2 \times \Sigma_3 \cdots \cup \Sigma_1 \times a_2 \times \Sigma_2 \cdots \in \mathcal{L}$, P3: $a_1 \times \Sigma_2 \times \Sigma_3 \cdots \in \mathcal{L} \Rightarrow a_1 \in \mathcal{L}_1 \cdots$ and P5: $(\exists\, j$ with $p_j \perp_j q_j) \Rightarrow (p_1, \cdots, p_n) \perp (q_1, \cdots, q_n)$.

2 Results

In the eighties, D. Aerts proposed a model for $\mathcal{L}_{ind}$, called the separated product [1]. His approach was to give explicitly, from Q_1 and Q_2, the set Q of all possible yes-no experiments on the total system. The separated product is defined as follows:

Definition 3 Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be cao-lattices.
(1) Let $p, q \in \Sigma_1 \times \Sigma_2$, $p \# q \Leftrightarrow p_1 \perp q_1$ or $p_2 \perp q_2$,
(2) $\mathcal{L}_1 \bigcirc\!\!\!\!\wedge\, \mathcal{L}_2 := \{a \subset \Sigma_1 \times \Sigma_2;\ a^{\#\#} = a\}$.

Remark 3 First $\mathcal{L}_1 \bigcirc\!\!\!\!\wedge\, \mathcal{L}_2$ is a cao-lattice [6]. Second, let $p \in \Sigma_1 \times \Sigma_2$, then $p^{\#} = p_1^{\perp_1} \times \Sigma_2 \cup \Sigma_1 \times p_2^{\perp_2}$.

In section 4, we will prove the following results (we say that $W_i \subset \mathrm{Aut}(\mathcal{L}_i)$ is transitive if $\forall p, q \in \Sigma_i$, $\exists\, u_i \in W_i$; $u_i(p) = q$, and $\mathcal{H}_i$ are Hilbert spaces over the complex numbers).

Theorem 1 *Let $\mathcal{L}_i = \mathrm{P}(\mathcal{H}_i)$ (with $\dim(\mathcal{H}_i) > 1$) and $\mathcal{L}$ be a cao-lattice. Let $U(\mathcal{H}_i)$ the group of unitary maps on $\mathcal{H}_i$. Then, $\mathcal{L}$ is P1, P2, P3 and P4 with $W_i = U(\mathcal{H}_i)$ $\Rightarrow \mathcal{L}$ does not have the covering property and $\mathcal{L}$ is not ortho-modular.*

Theorem 2 *Let $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}$ be cao-lattices. Suppose that $\mathrm{Aut}(\mathcal{L}_i)$ is transitive. Then, $\mathcal{L}$ is P1, P2, P3, P4 with W_i transitive and P5 $\Leftrightarrow \mathcal{L} = \mathcal{L}_1 \bigcirc\!\!\!\!\wedge\, \mathcal{L}_2$.*

Theorem 3 *Let $\mathcal{L}_i = \mathrm{P}(\mathcal{H}_i)$ and $\mathcal{L}$ be a cao-lattice. Then, $\mathcal{L}$ is P1, P2, P3 and (P4*) $u_1 \times u_2$ is an ortho-isomorphism of $\mathcal{L}$, $\forall\, u_i \in U(\mathcal{H}_i) \Leftrightarrow \mathcal{L} = \mathcal{L}_1 \bigcirc\!\!\!\!\wedge\, \mathcal{L}_2$.*

Finally, in section 5 we prove that for $\mathcal{L}_i = \mathrm{P}(\mathcal{H}_i)$, axioms P2, P3, P4 with $W_i = U(\mathcal{H}_i)$ and P5 are independent.

Theorem 1 asserts that no Hilbert space description is possible for two independent quantum systems. Aerts proved a similar result for the separated product [1] (see also [6]) and more generally for independent systems in [2] (see also [7]).

Assumption P4* may appear natural for u_i ortho-isomorphisms, but, again, its physical justification is delicate: if the ortho-complementation of $\mathcal{L}_{ind}$ is induced by (1), and if two final states are orthogonal at time t_1 then the yes-no experiment: "evolve the system from time t_0 to time t_1 and perform α" makes the two initial states at time t_0 orthogonal.

The separated product has been investigated in [6]. It is proved that $\mathcal{L}_1 \bigcirc\!\!\!\!\wedge\, \mathcal{L}_2$ is irreducible $\Leftrightarrow$ $\mathcal{L}_1$ and $\mathcal{L}_2$ are irreducible. Moreover, if $\mathcal{L}_1$ and $\mathcal{L}_2$ have the covering property, atomic endomorphisms (join-preserving maps sending atoms to atoms, that is evolution maps) preserve irreducible components and factor through the components: let $\mathcal{L}$ be an irreducible cao-lattice having the covering property and let f be an atomic endomorphism of $\mathcal{L} \textcircled{\wedge} \mathcal{L}$ with the image not contained in $\mathcal{L}$. Then there exist two atomic endomorphisms f^i of $\mathcal{L}$ and a permutation σ such that $f = \sigma(f^1 \times f^2)$ on the atoms.

3 Discussion and further questions

Consider two quantum systems described by $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$ (two q-bits). In ordinary quantum mechanics, the evolution is given by a unitary map u on $\mathbb{C}^2 \otimes \mathbb{C}^2$. If the two systems are initially independent, one always assumes the restriction

$$u : \Sigma_1 \times \Sigma_2 \to (\mathcal{H}_1 \otimes \mathcal{H}_2)^* / \mathbb{C}$$

to be the evolution map from time t_0 to time t_1 ($\Sigma_i = \mathbb{C}^{2*} / \mathbb{C}$). This assumption together with condition (2) imposes that $u^{-1}(V) \cap \Sigma_1 \times \Sigma_2 \in \mathcal{L}_{t_0}$ for any closed subspace V of $\mathbb{C}^2 \otimes \mathbb{C}^2$, that is $\mathcal{L}_0 := \{V \cap \Sigma_1 \times \Sigma_2;\ V \subset (\mathbb{C}^2 \otimes \mathbb{C}^2)^* / \mathbb{C}, V^{\perp_\otimes \perp_\otimes} = V\} \subset \mathcal{L}_{t_0}$, where $\perp_\otimes$ is the orthogonality relation in the tensor product.

In proposition 6 we prove that $\mathcal{L}_0 \supset \mathcal{L}_1 \textcircled{\wedge} \mathcal{L}_2$ but $\mathcal{L}_0 \neq \mathcal{L}_1 \textcircled{\wedge} \mathcal{L}_2$ (where $\mathcal{L}_i = \mathrm{P}(\mathbb{C}^2)$). Moreover, $\mathcal{L}_0$ has no ortho-complementation. As a consequence, if the above description of the interacting q-bits that are initially independent is imbued with physical reality, the property lattice of independent quantum systems $\mathcal{L}_{ind}$ is not the separated product. Moreover, in [6] we have proved that in the separated product, no model is possible for two interacting quantum systems that are independent before and after the interaction takes place. Remark that this shortcoming should for most physicists surprisingly not be an argument to exclude the separated product as a candidate for $\mathcal{L}_{ind}$ since ordinary two-body quantum theory excludes this situation.

Nevertheless, the separated product do not allow any interaction for quantum systems that are initially independent and therefore can be excluded. Thus, as a consequence of theorem 2, the assumption that relation (1) induces an ortho-complementation on $\mathcal{L}$ is wrong for independent quantum systems. The question that follows naturally is whether it is possible for independent quantum systems to assume that $\mathcal{L}$ is ortho-complemented. In proposition 5, we give an example of a cao-lattice $\mathcal{L}^5$ that satisfies properties P1 to P4 but not P5. But as a p-lattice, $\mathcal{L}^5$ is equal to the separated product. As a consequence, we propose the following question:

Question 1 Let $\mathcal{L}_1 = \mathcal{L}_2 = \mathrm{P}(\mathbb{C}^2)$. Does there exist a cao-lattice $\mathcal{L}$ that is P1, P2, P3 and P4 with $W_1 = W_2 = U(\mathbb{C}^2)$ and as a p-lattice $\mathcal{L} \neq \mathcal{L}_1 \textcircled{\wedge} \mathcal{L}_2$?

4 Proofs

Proof of Theorem 1: (1) Let $\mathcal{L}$ be a cao-lattice. Let $\mathcal{Z}(\mathcal{L})$ denote the center of $\mathcal{L}$, and for an atom p, let $\mathcal{Z}(p)$ be the central cover of p. Let $p \neq q$ be two atoms. Recall that if $\mathcal{L}$ has the covering property, then $\mathcal{Z}(p) = \mathcal{Z}(q) \Leftrightarrow p \vee q \neq \{p, q\}$ (see [6] or [8]).

Let $p, q \in \Sigma_1 \times \Sigma_2$ with $p_1 \neq q_1$ and $p_2 \neq q_2$. Then by P2 (we drop the subscripts $_1$ and $_2$ when no confusion can occur),

$$\begin{aligned} p \vee q &= [p_1 \times \Sigma \cap \Sigma \times p_2] \vee [q_1 \times \Sigma \cap \Sigma \times q_2] \\ &\subset [p_1 \times \Sigma \vee \Sigma \times q_2] \cap [q_1 \times \Sigma \vee \Sigma \times p_2] \\ &= [p_1 \times \Sigma \cup \Sigma \times q_2] \cap [q_1 \times \Sigma \cup \Sigma \times p_2] \\ &= \{p, q\} \ . \end{aligned} \tag{3}$$

Suppose that $\mathcal{L}$ has the covering property. Let p be an atom of $\mathcal{L}$. By (3), we can assume that $\mathcal{Z}(p) = r \times b$, where r is an atom and $b \subset \Sigma_2$. But by P4, for any $u \in W_1 \times W_2$, $u(\mathcal{Z}(p)) \in \mathcal{Z}(\mathcal{L})$, and therefore, since W_i are transitive, $r \times \Sigma \in \mathcal{Z}(\mathcal{L}), \forall r \in \Sigma_1$. Thus, by P3, $\mathcal{L}_1 = 2^{\Sigma_1}$, which is a contradiction.

(2) Suppose that $\mathcal{L}$ is ortho-modular. Then $p \vee q = \{p, q\} \Rightarrow\ p \perp q$ or $p = q$ (see [1] or [6]). Thus by (3), $p^\perp \supset p_1^c \times p_2^c$ (where $p_1^c = \Sigma_1 \backslash p_1$), $\forall p \in \Sigma$. Let $u \in W_1 \times W_2$, then by P4 $u(p^\perp)$ is a coatom and so $u(p^\perp) \supset u_1(p_1)^c \times u_2(p_2)^c \cup g_u(p)_1^c \times g_u(p)_2^c$ where $g_u(p) = u(p^\perp)^\perp$. As a consequence, $u(p^\perp) = u(p)^\perp\ \forall u \in W_1 \times W_2$ and $\forall p \in \Sigma$ (see the proof of theorem 2, part 1) and by theorem 3, $\mathcal{L} = \mathcal{L}_1 \circledwedge \mathcal{L}_2$. Finally, it is known that if $\mathcal{L}_1 \circledwedge \mathcal{L}_2$ is ortho-modular, then $\mathcal{L}_i = 2^{\Sigma_i}$ for $i = 1$ or 2 (see [1] or [6]). ∎

Proof of Theorem 2: $\underline{\Leftarrow}$: Let $a_i \subset \Sigma_i$. Then, by Definition 3,

$$(a_1 \times \Sigma)^{\#\#} = (a_1^\perp \times \Sigma)^{\#} = a_1^{\perp\perp} \times \Sigma$$

and

$$[a_1 \times \Sigma \cup \Sigma \times a_2]^{\#\#} = (a_1^\perp \times a_2^\perp)^{\#} = a_1^{\perp\perp} \times \Sigma \cup \Sigma \times a_2^{\perp\perp} \ .$$

Finally, put $W_i = \mathrm{Aut}(\mathcal{L}_i)$. Let $u_1 \times u_2 \in W_1 \times W_2$ and $p \in \Sigma$. Then

$$u_1 \times u_2(p^{\#}) = u_1(p_1^\perp) \times \Sigma \cup \Sigma \times u_2(p_2^\perp) \in \mathcal{L}_1 \circledwedge \mathcal{L}_2 \ .$$

$\underline{\Rightarrow}$: (0) Remark that P2 $\Rightarrow\ p^{\#} \in \mathcal{L},\ \forall p \in \Sigma$. (1) Let $p \in \Sigma$, define $p_{\#} := \{q \in \Sigma;\ q^{\#} \subset p^\perp\}$. Then by P3 and P4, $|p_{\#}| \leq 1$, $\forall p \in \Sigma$. Indeed, suppose that $|p_{\#}| > 1$. Let $p_{\#1} := \{q_1 \in \Sigma_1;\ q_1 \times \Sigma \subset p^\perp\}$ and, for $r \in \Sigma_1$,

$$C_r(p^\perp) := r \times \Sigma \cap p^\perp \backslash p_{\#1} \times \Sigma \ .$$

So

$$p^\perp = p_{\#1} \times \Sigma \bigcup_{r \in \Sigma_1 \backslash p_{\#1}} C_r(p^\perp) \ .$$

Suppose for instance that $p_{\#1} \notin \mathcal{L}_1$. Let $u_0 \in W_1$. We have

$$
\begin{aligned}
&\bigcap\{u_0 \times u(p^\perp);\ u \in W_2\} \\
&= \bigcap \left\{ u_0(p_{\#1}) \times \Sigma \cup \bigcup_{r\in\Sigma_1\backslash p_{\#1}} u_0(r) \times u(C_r(p^\perp)_2); u \in W_2 \right\} \\
&= u_0(p_{\#1}) \times \Sigma \bigcup_{r\in\Sigma_1\backslash p_{\#1}} u_0(r) \times \bigcap_{u\in W_2} u(C_r(p^\perp)_2).
\end{aligned}
$$

By definition, for any $r \in \Sigma_1 \backslash p_{\#1}$, $\exists s \in \Sigma_2;\ C_r(p^\perp)_2 \subset s^c := \Sigma_2\backslash s$. As a consequence, since by assumption W_2 is transitive,

$$\bigcap_{u\in W_2} u(C_r(p^\perp)_2) \subset \bigcap_{u\in W_2} u(s)^c = \emptyset\ .$$

By P4, we have that $u_0(p_{\#1}) \times \Sigma \in \mathcal{L}$, and by P3, $u_0(p_{\#1}) \in \mathcal{L}_1$, which is a contradiction since by assumption $p_{\#1} \notin \mathcal{L}_1$. Thus we have proved that

$$p^{\#\perp} \cap q^{\#\perp} = \emptyset,\ \forall p \neq q\ . \tag{4}$$

(2) By P5 $p^\# \subset p^\perp$, $\forall p \in \Sigma$, that is $p \in p^{\#\perp}$. As a consequence, (4) implies that $p^{\#\perp} = \{p\}$, $\forall p \in \Sigma$.

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Proof of Theorem 3: Denote $U(\mathcal{H}_i)$ by W_i. First, since W_i are transitive, P4*$\Rightarrow$ $\cup\{r^{\#\perp};\ r \in \Sigma\} = \Sigma$. Let $p \in q^{\#\perp}$ and let $G_p := \{u \in W_1 \times W_2;\ u(p) = p\}$. Then, by P4*, $p \in u(q)^{\#\perp}$, $\forall u \in G_p$, thus by (4), $u(q) = q$, $\forall u \in G_p$, that is, if $\dim(\mathcal{H}_i) \geq 3$ for $i = 1$ and 2, $q = p$ and by (4), $p^{\#\perp} = \{p\}$, $\forall p \in \Sigma$. The case where $\mathcal{H}_1 = \mathbb{C}^2$ or $\mathcal{H}_2 = \mathbb{C}^2$ is a simple extension.

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5 Examples with $\mathcal{L}_1 = \mathcal{L}_2 = \mathrm{P}(\mathbb{C}^2)$

Let $\mathcal{L}_1 = \mathcal{L}_2 = \mathrm{P}(\mathbb{C}^2)$ the lattice of subspaces of $\mathbb{C}^2$ and $U(\mathbb{C}^2)$ the group of unitary maps on $\mathbb{C}^2$. Then $\Sigma_i = \mathbb{C}^{2^*}/\mathbb{C}$. We give four examples $\mathcal{L}^2, \cdots \mathcal{L}^5$ of cao-lattices such that $\mathcal{L}^j$ satisfies properties P1 to P4 (with $W_i = W_i^j$) and P5 but not property Pj, where $W_i^2 = W_i^3 = U(\mathbb{C}^2)$, $W_i^5 = \mathrm{Aut}(\mathrm{P}(\mathbb{C}^2))$ and W_i^4 is transitive. Finally, we give an example of a p-lattice $\mathcal{L}_0$ that is not ortho-complemented and satisfies properties P1, P2, P3 and P4 with $W_i = \mathrm{Aut}(\mathrm{P}(\mathbb{C}^2))$ and $\mathcal{L}_0 \neq \mathcal{L}_1 \textcircled{\wedge} \mathcal{L}_2$.

$\mathcal{L}^j := \{a \subset \Sigma = \Sigma_1 \times \Sigma_2;\ a^{\perp_j\perp_j} = a\}$ where $\perp_j$ is an orthogonality relation, that is an anti-reflexive symmetric separating (i.e. $p^{\perp_j\perp_j} = p$, $\forall p \in \Sigma$) binary relation on Σ.

For $q \in \Sigma_i$, we denote $C(q) := \{r \in \Sigma_i;\ |\langle Q, R\rangle| = \sqrt{3}/2\}$ where $Q \in q$, $R \in r$ and $|Q| = |R| = 1$. Finally, remark that for $u \in U(\mathbb{C}^2)$, $u(C(q)) = C(u(q))$.

Lemma 1 *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be cao-lattices and $\mathcal{L}$ a p-lattice.*
(1) Suppose that $\mathcal{L}$ is P1. Then $\mathcal{L}$ is P2 $\Leftrightarrow p^\# \in \mathcal{L}$, $\forall p \in \Sigma$.
(2) Let $\perp$ be an anti-reflexive symmetric binary relation on $\Sigma = \Sigma_1 \times \Sigma_2$. If $p^{\#\perp\perp} \subset p^\#$, $\forall p \in \Sigma$, then $\perp$ is an orthogonality relation.

Proof : (1) $\Rightarrow$: follows from P2 by definition. $\Leftarrow$: If $p^\# \in \mathcal{L}$, $\forall p \in \Sigma$, then $\mathcal{L}_1 \textcircled{\scriptsize$\wedge$} \mathcal{L}_2 \subset \mathcal{L}$, so by Theorem 2, $\mathcal{L}$ is P2.

(2) Let $p \in \Sigma$, then

$$p^{\perp\perp} = (\cap\{q^{\#\perp\perp};\ p \in q^\#\})^{\perp\perp} = \cap\{q^\#;\ p \in q^\#\} = p\ ,$$

because for any $a \subset \Sigma$, $a^{\perp\perp\perp} = a^\perp$.

■

Lemma 2 *Let $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}$ be cao-lattices. Suppose that $\mathcal{L}$ is P1. If $p_{\#i} \in \mathcal{L}_i$, $\forall p \in \Sigma$ and $i = 1, 2$, then $\mathcal{L}$ is P3 (where $p_{\#i}$ is defined in the proof of Theorem 2).*

Proof : Since by assumption $p_{\#1} \in \mathcal{L}_1$, $\forall p \in \Sigma$, if $a \times \Sigma = \cap\{p^\perp;\ a \times \Sigma \subset p^\perp\}$, then $a \in \mathcal{L}_1$.

■

Proposition 2 *For $p \in \Sigma = \Sigma_1 \times \Sigma_2$, define $p^{\perp_2} := p^\# \cup C(p_1) \times C(p_2)$. Then $\mathcal{L}^2$ is a cao-lattice and $\mathcal{L}^2$ is P3, P4 with $W_i = U(\mathbb{C}^2)$ and P5 but not P2.*

Proof : (1) We check that $\perp_2$ is an orthogonality relation: (i) By definition, $\perp_2$ is anti-reflexive. (ii) Since $q \in C(p) \Rightarrow p \in C(q)$, $\perp_2$ is symmetric. (iii) Finally

$$(p_1^\perp \times \Sigma)^{\perp_2} = \bigcap_{q \in \Sigma_2} (p_1^\perp, q)^{\perp_2} \subset \bigcap_{q \in \Sigma_2} p_1 \times \Sigma \cup \Sigma \times q^c = p_1 \times \Sigma\ , \tag{5}$$

thus $p^{\perp_2\perp_2} = p$, $\forall p \in \Sigma$.

(2) By definition, $\mathcal{L}^2$ is P5 and P4 with $W_i = U(\mathbb{C}^2)$, since $u(p^{\perp_2}) = u(p)^\# \cup C(u_1(p_1)) \times C(u_2(p_2)) = u(p)^{\perp_2}$. By lemma 2, $\mathcal{L}^2$ is P3.

(3) Finally, $\mathcal{L}^2$ is not P2 because, $p^\# \notin \mathcal{L}^2$. Indeed, by (5), $p^{\#\perp_2} = p$ so that $p^{\#\perp_2\perp_2} = p^{\perp_2} \neq p^\#$. Remark that $\mathcal{L}^2$ is not P2 also as a consequence of Theorem 2 or Theorem 3.

■

Proposition 3 *For $p \in \Sigma = \Sigma_1 \times \Sigma_2$, define $p^{\perp_3} := p^\# \cup C(p_1) \times \Sigma_2 \cup \Sigma_1 \times C(p_2)$. Then $\mathcal{L}^3$ is a cao-lattice and $\mathcal{L}^3$ is P2, P4 with $W_i = U(\mathbb{C}^2)$ and P5 but not P3.*

Proof : (1) For the same reasons as in Proposition 2, $\perp_3$ is anti-reflexive and symmetric and $\mathcal{L}^3$ is P4 with $W_i = U(\mathbb{C}^2)$. By part (2) and Lemma 1, $\perp_3$ is an orthogonality relation. By definition, $\mathcal{L}^3$ is P5.

(2) $\mathcal{L}^3$ is P2: Indeed, let $\Omega = C(p_1^\perp) \times C(p_2^\perp)$, then

$$\begin{aligned} p^{\#\perp_3\perp_3} &\subset \bigcap\{q^{\perp_3} = (q_1^\perp \cup C(q_1)) \times \Sigma \cup \Sigma \times (q_2^\perp \cup C(q_2));\ q \in \Omega\} \\ &= \bigcup_{\omega \subset \Omega} \bigcap_{q \in \omega} \{q_1^\perp \cup C(q_1)\} \times \bigcap_{r \in \Omega \setminus \omega} \{r_2^\perp \cup C(r_2)\} = p^\#\ , \end{aligned}$$

since if $q \neq r \neq s \neq q \in C(p_i^\perp)$, then $C(q) \cap C(r) \cap C(s) = \{p_i^\perp\}$.

(3) By definition, $|p_{\#}| > 1$, thus $\mathcal{L}^3$ is not P3 (see the proof of Theorem 2, part 1). Remark that $\mathcal{L}^3$ is not P3 also as a consequence of Theorem 2 or Theorem 3.

■

Proposition 4 *Let $A_i \subset \Sigma_1$ and $g_i : A_i \to \Sigma = \Sigma_1 \times \Sigma_2$ be bijections ($i = 1, 2, 3, 4$) such that $A_i \cap A_j = \emptyset$ and $A_1 \cup \cdots \cup A_4 = \Sigma_1$. Suppose moreover that $\forall i$,*

$$g_i(p) \cap (p,p)^{\#} = \emptyset \ \forall p \in A_i \ . \tag{6}$$

Define $f : \Sigma \to \Sigma \cup \{\Sigma\}$ by $f(p) = \Sigma$ if $p_1 \neq p_2$ and $f(p,p) = g_i(p)$ if $p \in A_i$. Put

$$p^{\perp_4} := p^{\#} \cup f(p)^{\#} \cup f^{-1}(p^{\#}) \ .$$

Then, $\mathcal{L}^4$ is a cao-lattice, $\mathcal{L}^4$ is P2, P3 and P5 but not P4 for any transitive W_i.

Proof : We want $\mathcal{L}^4$ to be P2, P3 and P5 but different from $\mathcal{L}_1 \textcircled{\wedge} \mathcal{L}_2$ so that by Theorem 2, $\mathcal{L}^4$ is not P4. By P5, $p^{\perp_4} = p^{\#} \cup a_p$ where $a_p \subset \Sigma$. So $a_p \neq \emptyset$ at least for one p. Since we want $\mathcal{L}^4$ to be P2, there must be at least one $r \neq p$ with $p^{\#} \subset r^{\perp_4}$. Further, since $\perp_4$ must be symmetric, $r \in (p_1^{\perp}, y)^{\perp_4}$, $(x, p_2^{\perp})^{\perp_4}$ for any x and y. In this example we choose to add one additional coatom of the separated product only to the ortho-complement of symmetric atoms (*i.e.* of the form (p,p)).

(1) By assumption (6), $\perp_4$ is anti-reflexive. Further, let $q \in f(p)^{\#}$, then $p \in f^{-1}(q^{\#})$ and if $q \in f^{-1}(p^{\#})$ then $p \in f(q)^{\#}$. Thus $\perp_4$ is symmetric.

(2) $\mathcal{L}^4$ is P2: $p^{\#} \subset q^{\perp_4} \Leftrightarrow q = p$ or $q \in f^{-1}(p)$. Let $\omega \subset f^{-1}(p)$ (note that $|f^{-1}(p)| = 4$). Since the atoms in ω are symmetric, if w has more than two elements, then

$$\bigcap_{q \in \omega} q^{\#} = \emptyset \text{ thus } \bigcap_{q \in \omega} f^{-1}(q^{\#}) = f^{-1}(\cap_{q\in\omega} q^{\#}) = \emptyset \ .$$

Moreover, since the inverse image by f of an atom contains only symmetric atoms,

$$\bigcap_{q \in \omega} f^{-1}(q^{\#}) \bigcap_{q \in f^{-1}(p) \setminus \omega} q^{\#} = \emptyset$$

if ω has two elements. As a consequence,

$$\bigcap_{q \in f^{-1}(p)} q^{\perp_4} = p^{\#}, \ \forall p \in \Sigma \ .$$

Thus, by Lemma 1, $\mathcal{L}^4$ is a cao-lattice and $\mathcal{L}^4$ is P2.

(3) $\mathcal{L}^4$ is P3: let $r^{\perp} \neq s^{\perp} \in \Sigma_1$. Then $\{r^{\perp}, s^{\perp}\} \times \Sigma \subset q^{\perp_4} \Leftrightarrow (q = (r,r)$ and $f(r)_1 = s)$ or $(q = (s,s)$ and $f(s)_1 = r)$. But

$$f^{-1}((r,r)^{\#}) \cap f^{-1}((s,s)^{\#}) \supset f^{-1}(r^{\perp}, s^{\perp}) \neq \emptyset$$

thus $\{r^{\perp}, s^{\perp}\} \times \Sigma \notin \mathcal{L}^4$.

(4) By definition, $\mathcal{L}^4$ is P5 and $\mathcal{L}^4 \neq \mathcal{L}_1 \bigotimes \mathcal{L}_2$, so that by Theorem 2, $\mathcal{L}^4$ is not P4 for any transitive W_i.

■

Example 1 Let $h : \Sigma_1 \to \Sigma_1$ be defined by $h(p) = p^\perp$. Let $E_1, \cdots E_4 \subset \Sigma_1$ (non countable) with $E_i \cap E_j = \emptyset$ if $i \neq j$, $\cup_{i=1}^4 E_i = \Sigma_1$ and $h(E_j) = E_{(j+2)}$ where $(.) = . - 1 \bmod 4 + 1$.

Let $A_i^k \subset \Sigma_1$ $(i, k = 1 \cdots 4)$ with $A_i^k \cap A_j^k = \emptyset$ if $i \neq j$ and $\cup_{i=1}^4 A_i^k = E_k$. Moreover let

$$g_i^k : A_i^k \to E_k \times E_k \cup E_k \times E_{(k+1)} \cup E_{(k+1)} \times E_k \cup E_{(k+1)} \times E_{(k+3)}$$

be bijections.

Define $A_i = \cup_{k=1}^4 A_i^k$ and $g_i : A_i \to \Sigma$ by $g_i(p) = g_i^k(p)$ if $p \in A_i^k$. Then $\{(A_i, g_i)\}_{i=1\cdots4}$ satisfies all conditions of Proposition 4.

Proposition 5 *Let f be a bijection of Σ_1 with*
(i) $f \neq id$
(ii) $f^{-1}(p^\perp) = f(p)^\perp$
(iii) $f(p) \neq p^\perp$, $\forall p \in \Sigma_1$. Define $p^{\perp_5} = (f \times f)(p)^\#$ $\forall p \in \Sigma$. Then $\mathcal{L}^5$ is a cao-lattice and $\mathcal{L}^5$ is P2, P3 and P4 with $W_i = \mathrm{Aut}(\mathrm{P}(\mathbb{C}^2))$, but $\mathcal{L}^5$ is not P5.

Proof : (1) By assumption (iii), $\perp_5$ is anti-reflexive and by assumption (ii), $\perp_5$ is symmetric. Moreover, since $f \times f$ is bijective, by Lemma 1, $\mathcal{L}^5$ is a cao-lattice and $\mathcal{L}^5$ is P2. As a p-lattice, $\mathcal{L}^5 = \mathcal{L}_1 \bigotimes \mathcal{L}_2$, so that $\mathcal{L}^5$ is P3 and P4 with $W_i = \mathrm{Aut}(\mathrm{P}(\mathbb{C}^2))$.

(2) By assumption (i), $\mathcal{L}^5$ is not P5.

■

Example 2 For $q \in \Sigma_1$, write $q = \mathbb{C}(r, c(r)\mathrm{e}^{\mathrm{i}\theta})$, with $r \in [0, 1]$ and $c(r) = \sqrt{1 - r^2}$. Define for $r \neq 0, 1$ and $\theta \in [0, \pi[$, $f(q) := \mathbb{C}(r^2, c(r^2)\mathrm{e}^{\mathrm{i}\theta})$, for $\theta \in [\pi, 2\pi[$, $f(q) := \mathbb{C}(\sqrt{1 - c(r)}, (1 - r^2)^{1/4}\mathrm{e}^{\mathrm{i}\theta})$, $f(\mathbb{C}(1, 0)) := \mathbb{C}(1, 0)$ and $f(\mathbb{C}(0, 1)) := \mathbb{C}(0, 1)$. Then f is a bijection that satisfies conditions (i) to (iii).

Proposition 6 *Let $\mathcal{L}_0 := \{V \cap \Sigma = \Sigma_1 \times \Sigma_2;\ V \subset (\mathbb{C}^2 \otimes \mathbb{C}^2)^* / \mathbb{C}, V^{\perp_\otimes \perp_\otimes} = V\}$ where $\perp_\otimes$ is the orthogonality relation in the tensor product. Then $\mathcal{L}_0$ is a p-lattice, $\mathcal{L}_0$ is P1, P2, P3 and P4 with $W_i = \mathrm{Aut}(\mathrm{P}(C^2))$ and $\mathcal{L}_0$ is not ortho-complemented.*

Proof : (1) $a \in \mathcal{L}_0 \Leftrightarrow a \in \mathcal{L}_1 \bigotimes \mathcal{L}_2$ or $a = \{p, q, r\}$ with $p, q, r \in \Sigma$ and $p_i \neq q_i \neq r_i \neq p_i$ for $i = 1, 2$. First, $a \in \mathcal{L}_1 \bigotimes \mathcal{L}_2 \Rightarrow a = \emptyset,\ \Sigma,\ p,\ p^\#,\ p_1 \times \Sigma,\ \Sigma \times p_2$ or $\{p, q\}$ where $p, q \in \Sigma$, $p_1 \neq q_1$ and $p_2 \neq q_2$ [6].

Let $V \subset (\mathbb{C}^2 \otimes \mathbb{C}^2)^* / \mathbb{C}$ with $V^{\perp_\otimes \perp_\otimes} = V$, and $x = V \cap \Sigma$. Let $k = \dim(\langle x \rangle)$ where $\langle x \rangle$ is the subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$ spanned by x and let $a = \{p^1, \cdots p^k \in x\}$ be a basis of $\langle x \rangle$. Remark that since $x^{\perp_\otimes} = a^{\perp_\otimes}$, $x^\# = a^\#$ and so $x^{\#\#} = a^{\#\#}$. If $k = 1$, then $x \in \mathcal{L}_1 \bigotimes \mathcal{L}_2$. If $k = 2$, then either $p_1^1 = p_1^2$ or $p_2^1 = p_2^2$ or $p_i^1 \neq p_i^2$ for $i = 1, 2$. As a consequence, $a^{\#\#} \subset x$ and so $x \in \mathcal{L}_1 \bigotimes \mathcal{L}_2$. Finally, if $k = 3$, either $a^{\#\#}$ is a coatom of $\mathcal{L}_1 \bigotimes \mathcal{L}_2$ and so $a^{\#\#} \subset x$, or $p_i^1 \neq p_i^2 \neq p_i^3 \neq p_i^1$ for $i = 1, 2$. But then $x = \{p^1, p^2, p^3\}$.

(2) $\mathcal{L}_0$ is not ortho-complemented: For any $p, q, r \in \Sigma$ with $p_i \neq q_i \neq r_i \neq p_i$ for $i = 1, 2$, Σ covers $\{p, q, r\}$. Suppose that $\mathcal{L}_0$ is ortho-complemented, note $r^\perp = \{p^1, q^1, r^1\}$ and $s^\perp = \{p^2, q^2, r^2\}$ with $\{p^1, q^1, r^1\} \cap \{p^2, q^2, r^2\} = \emptyset$. Then $r \vee s = \Sigma$ what is a contradiction (see part 1).

■

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